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Coupling of Quantum Logics

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A quantum logic is a couple (L, M), where L is a logic and M is a quite full set of states on L. A tensor product in the category of quantum logics is defined and a comparison with the definition of free orthodistributive product of orthomodular σ lattices is given. Several physically important cases are treated.

1. INTRODUCTION

The problem of coupling of logics was treated by several authors (Aerts, 1979; Aerts and Daubechies, 1978; Matolcsi, 1975; Zecca, 1978, 1979). It is supposed that the logic L of a physical system S, which is composed of two physical systems S_1 and S_2 with the logics L_1 and L_2, respectively, is a kind of tensor product (or free orthodistributive product) of the logics L_1 and L_2. Essentially, only the case in which the logics were complete and atomistic orthomodular lattices was treated. In the category of Hilbert space logics, there was shown (Matolcsi, 1975; Aerts and Daubechies, 1978) that there are two tensor products of the logics $L_1(H_1)$ and $L_2(H_2)$, namely, $L(H_1 \otimes H_2)$, i.e., the logic of the tensor product $H_1 \otimes H_2$, and $L(\overline{H}_1 \otimes H_2)$, i.e., the logic of the tensor product $\overline{H}_1 \otimes H_2$, where $\overline{H}_1$ is the dual of H_1. [The case of real or complex separable Hilbert spaces of the dimension at least three was considered. In the case of complex Hilbert spaces the tensor products $L(H_1 \otimes H_2)$ and $L(\overline{H}_1 \otimes H_2)$ are not equivalent.]

The definition of a tensor product (or free orthodistributive product) of orthomodular σ lattices was proposed by Matolcsi (1975) in the following form.

Definition 1. Let $L_i (i \in I)$ and L be orthomodular σ lattices. Then $(L, (u_i)_{i \in I})$ is a tensor product (or free orthodistributive product) of the L_is

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if (i) $u_i: L_i \to L$ are orthoinjections ($i \in I$), (ii) $\cup_{i \in I} u_i(L_i)$ generates L, (iii) for every finite or countable subset F of I, $\cup_{i \in F} u_i(a_i) = 0$ for $a_i \in L_i$ if and only if at least one a_i is zero, and (iv) $u_i(a_i)$ is compatible with $u_j(a_j)$ for all $i, j \in I$ such that $i \neq j$.

2. TENSOR PRODUCT OF QUANTUM LOGICS

In this paper, we shall call a "quantum logic" the pair (L, M), where L is an orthomodular σ lattice (we shall call it a logic) and M is a set of states which is quite full for L, i.e.,

$$\{m \in M: m(a) = 1\} \subset \{m \in M: m(b) = 1\} \qquad \text{implies } a = b$$

$$(a, b \in L) \quad (1)$$

We shall further suppose that the Jauch–Piron condition in the countable form is satisfied, i.e.,

$$m(a_i) = 1 \quad \text{for all } i = 1, 2, \ldots \quad \text{implies } m\left(\bigwedge_{i=1}^{\infty} a_i\right) = 1 \qquad (m \in M) \quad (2)$$

Basic facts on logics and states can be found in Varadarajan (1968).

We shall give a definition of the tensor product in the category of quantum logics. The definition is given for two quantum logics (L_1, M_1) and (L_2, M_2), but it can be in a natural way generalized to any set (L_i, M_i), $i \in I$.

Definition 2. Let $(L_1, M_1), (L_2, M_2), (L, M)$ be quantum logics. We say that (L, M) is a tensor product of (L_1, M_1) and (L_2, M_2) if there are mappings α, β such that:

(i) $\alpha: L_1 \times L_2 \to L$, $\beta: M_1 \times M_2 \to M$,

$$\beta(m_1, m_2)(\alpha(a_1, a_2)) = m_1(a_1) m_2(a_2)$$

for any $m_i \in M_i$, $a_i \in L_i$, $i = 1, 2$. Here $L_1 \times L_2$ and $M_1 \times M_2$ are the direct products.

(ii) $\beta[M_1 \times M_2] = \{\beta(m_1, m_2): m_1 \in M_1, m_2 \in M_2\}$ is quite full for L.

(iii) L is generated by $\alpha[L_1 \times L_2]$, i.e., the smallest sublogic of L containing all $\alpha(a_1, a_2)$, $a_1 \in L_1$, $a_2 \in L_2$, is L.

We shall denote the product by $(L, M)_{\alpha, \beta}$.

Let L_1, L_2 be orthomodular σ lattices. A map $\psi: L_1 \to L_2$ is a σ orthohomomorphism if (i) $\psi(1) = 1$, (ii) $\psi(\vee_{i=1}^{\infty} a_i) = \vee_{i=1}^{\infty} \psi(a_i)$ for any sequence $(a_i) \subset L_1$, (iii) $\psi(a^{\perp}) = \psi(a)^{\perp}$, $a \in L_1$.

A σ orthohomomorphism is called orthoinjection if it is one-to-one. A σ orthohomomorphism which is one-to-one and onto is a bijection.

Proposition 1. Let us define

$$\varphi_1: L_1 \to L, \qquad \varphi_2: L_2 \to L$$
$$a_1 \mapsto \alpha(a_1, 1) \qquad a_2 \mapsto \alpha(1, a_2)$$

Then φ_1, φ_2 are orthoinjections.

Proof. From $\beta(m_1, m_2)(\alpha(1,1)) = m_1(1)m_2(1) = 1$ for all $m_1 \in M_1, m_2 \in M_2$, and from the fact that $\beta[M_1 \times M_2]$ is quite full for L, we obtain that $\alpha(1,1) = 1$. (We write 1 for the greatest element in any of L_1, L_2, L). From this we have that $\varphi_1(1) = 1$, $\varphi_2(1) = 1$. Further,

$$\begin{aligned}\beta(m_1, m_2)\big(\alpha(a_1{}^{\perp}, 1)\big) &= m_1(a_1{}^{\perp})m_2(1) \\ &= m_1(a_1^{\perp}) = 1 - m_1(a_1) = \big(1 - m_1(a_1)\big)m_2(1) \\ &= 1 - m_1(a_1)m_2(1) = 1 - \beta(m_1, m_2)\big(\alpha(a_1, 1)\big) \\ &= \beta(m_1, m_2)\big(\alpha(a_1, 1)^{\perp}\big)\end{aligned}$$

for all $m_1 \in M_1, m_2 \in M_2$, which implies that $\varphi_1(a_1{}^{\perp}) = \varphi_1(a_1)^{\perp}$. Similarly, $\varphi_2(a_2{}^{\perp}) = \varphi_2(a_2)^{\perp}$. Now let $(a_1{}^k)_{k=1}^{\infty}$ be any sequence in L_1. From the Jauch–Piron property (1) we get $\beta(m_1, m_2)(\alpha(\wedge_k a_1{}^k, 1)) = 1$ iff $m_1(\wedge_k a_1{}^k) = 1$ iff $m_1(a_1{}^k) = 1$ for all k iff $\beta(m_1, m_2)(\wedge_k \alpha(a_1{}^k, 1)) = 1$ for any $m_1 \in M_1, m_2 \in M_2$, which implies that $\alpha(\wedge_k a_1{}^k, 1) = \wedge_k \alpha(a_1{}^k, 1)$, i.e., $\varphi_1(\wedge_k a_1{}^k) = \wedge_k \varphi_1(a_1{}^k)$. By the duality we obtain that $\varphi_1(\vee_k a_1{}^k) = \vee_k \varphi_1(a_1{}^k)$, so that φ_1 is a σ orthohomomorphism. The same holds for φ_2. Now $\beta(m_1, m_2)(\alpha(a_1, 1)) = \beta(m_1, m_2)(\alpha(a_1', 1))$ for all $m_1 \in M_1, m_2 \in M_2$ implies that $m_1(a_1) = m_1(a_1')$ for all $m_1 \in M_1$, so that $a_1 = a_1'$. From this we see that φ_1 and φ_2 are injections. ■

Proposition 2. For any $a_1 \in L_1$ and $a_2 \in L_2$, $\varphi_1(a_1)$ is compatible with $\varphi_2(a_2)$.

Proof. For any $m_1 \in M_1, m_2 \in M_2$,

$$\beta(m_1, m_2)(\alpha(a_1, 1) \wedge \alpha(1, a_2)) = 1 \quad \text{iff } \beta(m_1, m_2)(\alpha(a_1, 1)) = 1,$$
$$\beta(m_1, m_2)(\alpha(1, a_2)) = 1 \quad \text{iff } m_1(a_1) = 1, m_2(a_2) = 1 \text{ iff}$$
$$\beta(m_1, m_2)(\alpha(a_1, a_2)) = 1$$

which implies that $\alpha(a_1, 1) \wedge \alpha(1, a_2) = \alpha(a_1, a_2)$. Now

$$\beta(m_1, m_2)(\alpha(a_1, 1) \wedge \alpha(1, a_2)) = 1 \quad \text{iff } \beta(m_1, m_2)(\alpha(a_1, 1)) = 1,$$
$$\beta(m_1, m_2)(\alpha(1, a_2)) = 1 \quad \text{iff } m_1(a_1) = 1, m_2(a_2) = 1 \text{ iff}$$
$$\beta(m_1, m_2)(\alpha(a_1, a_2)) = 1$$

for any $m_1 \in M_1, m_2 \in M_2$, implies that $\varphi_1(a_1)$ and $\varphi_2(a_2)$ are independent (in the probabilistic sense) in all states of $\beta[M_1 \times M_2]$. This implies, in particular, that $\varphi_1(a_1)$ and $\varphi_2(a_2)$ are compatible (see Gudder, 1968). ■

Theorem 1. Let $(L, M)_{\alpha,\beta}$ be the tensor product of (L_1, M_1) and (L_2, M_2) in the sense of Definition 2. If we put

$$\varphi_1: L_1 \to L, \qquad \varphi_2: L_2 \to L$$
$$a_1 \mapsto \alpha(a_1, 1) \qquad a_2 \mapsto \alpha(1, a_2)$$

then $(L, \varphi_1, \varphi_2)$ is the tensor product of L_1 and L_2 in the sense of Definition 1.

Proof. (i) Evidently, $u_1(L_1) \cup u_2(L_2) \subset [u_1(L_1) \cup u_2(L_2)]'' \subset L'' = L$. On the other hand, $[u_1(L_1) \cup u_2(L_2)]''$ is an orthomodular sub-σ-lattice of L, containing both $u_1(L_1)$ and $u_2(L_2)$. As L is generated by $u_1(L_1)$ and $u_2(L_2)$, we obtain that $[u_1(L_1) \cup u_2(L_2)]'' = L$.

To prove (i), let $\varphi_1(a_1) \wedge \varphi_2(a_2) = 0$ and $a_1 \neq 0$. As $\varphi_1(a_1) \wedge \varphi_2(a_2) = \alpha(a_1, a_2)$, we get from $\beta(m_1, m_2)(\varphi_1(a_1) \wedge \dot{\varphi}_2(a_2)) = 0$ for any $m_1 \in M_1, m_2 \in M_2$, that $m_1(a_1)m_2(a_2) = 0$ for any $m_1 \in M_1, m_2 \in M_2$. Let $m_1{}^0 \in M_1$ be such that $m_1{}^0(a_1) = 1$. (Such $m_1{}^0$ exists because M_1 is quite full for L_1 and $a_1 \neq 0$.) Then $m_1{}^0(a_1)m_2(a_2) = 0$ for any $m_2 \in M_2$ implies that $m_2(a_2) = 0$ for any $m_2 \in M_2$, i.e., $a_2 = 0$.

(ii) By Definition 2 (iii), L is generated by $\alpha[L_1 \times L_2]$. As for any $a_1 \in L_1, a_2 \in L_2$, $\alpha(a_1, a_2) = \varphi_1(a_1) \wedge \varphi_2(a_2)$, we see that $\varphi_1(L_1) \cup \varphi_2(L_2)$ generates L. ■

3. SOME PROPERTIES OF THE TENSOR PRODUCT

Let (L, u_1, u_2) be the free orthodistributive product of L_1 and L_2 in the sense of Definition 1. For a subset M of an orthomodular lattice K put $M' = \{a \in K: a \leftrightarrow b \text{ for any } b \in M\}$. (We write $a \leftrightarrow b$ if a is compatible with b.) The set K' is the center of K. We shall study the relations between the centers L_1', L_2', and L'. We shall need the following lemma.

Lemma 1.A. σ homomorphism $u: L_1 \to L_2$ between two orthomodular σ lattices L_1, L_2 is injective iff $u(a) = 0$ implies $a = 0$ $(a \in L_1)$.

Proof. Let $u(a) = 0$ imply $a = 0$ and let $u(a) \leqslant u(b)$, $a, b \in L_1$. Then $u(a) - u(a \wedge b) = 0$ implies $u(a - a \wedge b) = 0$ and this implies $a - a \wedge b = 0$, i.e., $a = a \wedge b$. Hence, $u(a) \leqslant u(b)$ implies $a \leqq b$. From this it follows that u is injective. The converse implication is clear. ∎

Theorem 2. Let (L, u_1, u_2) be the free orthodistributive product of L_1 and L_2 in the sense of Definition 1. Then the following hold:

(i) $[u_1(L_1) \cup u_2(L_2)]'' = L$

(ii) $[u_1(L_1) \wedge u_2(L_2)] \cap [u_1(L_1) \wedge u_2(L_2)]' = u_1(L_1') \wedge u_2(L_2')$

where $K_1 \wedge K_2 = \{a \wedge b: a \in K_1, b \in K_2\}$, K_1 and K_2 are any lattices, and

(iii) $[u_1(L_1') \cup u_2(L_2')]'' = L'$

Proof. (i) Evidently, $u_1(L_1) \cup u_2(L_2) \subset [u_1(L_1) \cup u_2(L_2)]'' \subset L'' = L$. On the other hand, $[u_1(L_1) \cup u_2(L_2)]''$ is an orthomodular sub-σ-lattice of L, containing both $u_1(L_1)$ and $u_2(L_2)$. As L is generated by $u_1(L_1)$ and $u_2(L_2)$, we obtain that $[u_1(L_1) \cup u_2(L_2)]'' = L$.

(ii) As $a \leftrightarrow b$, $a, b \in L_1$, implies $u_1(a) \leftrightarrow u_1(b)$ in L, we have $u_1(L_1') \subset u_1(L_1)'$. By Definition 1 (iv), $u_2(L_2) \subset u_1(L_1)'$ and $u_1(L_1) \subset u_2(L_2)'$. Evidently, $u_1(L_1') \subset u_1(L_1)$. Now if $a \in u_1(L_1') \wedge u_2(L_2')$ is of the form $a = u_1(a_1) \wedge u_2(a_2)$, then $u_1(a_1) \leftrightarrow u_1(L_1)$ [i.e., $u_1(a_1) \leftrightarrow u_1(b_1)$ for any $b_1 \in L_1$] and $u_1(a_1) \leftrightarrow u_2(L_2)$, from which it follows that $u_1(a_1) \leftrightarrow u_1(L_1) \wedge u_2(L_2)$. Similarly, $u_2(a_2) \leftrightarrow u_1(L_1) \wedge u_2(L_2)$. From this it follows that $u_1(a_1) \wedge u_2(a_2) \in [u_1(L_1) \wedge u_2(L_2)]'$. Hence $u_1(L_1') \wedge u_2(L_2') \subset [u_1(L_1) \wedge u_2(L_2)]' \cap u_1(L_1) \wedge u_2(L_2)$.

On the other hand, let $a \in [u_1(L_1) \wedge u_2(L_2)]' \cap u_1(L_1) \wedge u_2(L_2)$ be of the form $a = u_1(a_1) \wedge u_2(a_2)$ $(a_1 \in L_1, a_2 \in L_2)$. We have $a \leftrightarrow u_1(L_1) \wedge u_2(L_2)$, especially $a \leftrightarrow u_1(b_1)$ for all $b_1 \in L_1$ and $a \leftrightarrow u_2(b_2)$ for all $b_2 \in L_2$.

Thus

$$u_1(b_1) = [u_1(a_1) \wedge u_2(a_2)] \wedge u_1(b_1) \vee [u_1(a_1) \wedge u_2(a_2)]^{\perp} \wedge u_1(b_1)$$
$$= [u_1(a_1) \wedge u_2(a_2)] \wedge u_1(b_1) \vee \left[u_1(a_1)^{\perp} \vee u_2(a_2)^{\perp}\right] \wedge u_1(b_1)$$

and

$$\begin{aligned} u_1(b_1) \wedge u_2(a_2) &= u_1(a_1) \wedge u_2(a_2) \wedge u_1(b_1) \vee \left[u_1(a_1)^{\perp} \vee u_2(a_2)^{\perp}\right] \\ &\quad \wedge u_2(a_2) \wedge u_1(b_1) \\ &= u_1(a_1 \wedge b_1) \wedge u_2(a_2) \vee u_1(a_1^{\perp} \wedge b_1) \wedge u_2(a_2) \\ &= u_1(a_1 \wedge b_1 \vee a_1^{\perp} \wedge b_1) \wedge u_2(a_2). \end{aligned} \tag{3}$$

Now let us consider the map

$$u_{1,a_2}: L_1 \to L$$
$$a_1 \mapsto u_1(a_1) \wedge u_2(a_2)$$

where $0 \neq a_2 \in L_2$ is fixed. As $u_1(a_1) \leftrightarrow u_2(a_2)$ for all $a_1 \in L_1$, u_{1,a_2} is a σ orthohomomorphism from L_1 into $L_{[0,u_2(a_2)]} = \{b \in L : b \leqslant u_2(a_2)\}$. By Lemma 1, u_{1,a_2} is injective. From this it follows that (3) implies that $b_1 = a_1 \wedge b_1 \vee a_1^{\perp} \wedge b_1$ for any $b_1 \in L_1$, hence $a_1 \in L_1'$. Similarly, $a_2 \in L_2'$. Thus we have shown that

$$[u_1(L_1) \wedge u_2(L_2)]' \cap u_1(L_1) \wedge u_2(L_2) \subseteq u_1(L_1') \wedge u_2(L_2')$$

(iii) For any $A, B \subset L$ we have $(A \cap B)' \supset (A' \cup B')''$. By (ii) we get

$$\begin{aligned} [u_1(L_1') \wedge u_2(L_2')]' &= (u_1(L_1) \wedge u_2(L_2) \cap [u_1(L_1) \wedge u_2(L_2)]')' \\ &\supset ([u_1(L_1) \wedge u_2(L_2)]' \cup [u_1(L_1) \wedge u_2(L_2)]'')'' \end{aligned}$$

As $u_1(L_1) \subset u_1(L_1) \wedge u_2(L_2)$, $u_2(L_2) \subset u_1(L_1) \wedge u_2(L_2)$, we have

$$[u_1(L_1) \cup u_2(L_2)]'' \subset [u_1(L_1) \wedge u_2(L_2)]''$$

On the other hand, as $a \leftrightarrow b_1, a \leftrightarrow b_2$ imply $a \leftrightarrow b_1 \wedge b_2$, $a, b_1, b_2 \in L$, we get

$$[u_1(L_1) \cup u_2(L_2)]' \subset [u_1(L_1) \wedge u_2(L_2)]'$$

i.e.,

$$[u_1(L_1) \cup u_2(L_2)]'' = [u_1(L_1) \wedge u_2(L_2)]''$$

Hence

$$([u_1(L_1) \wedge u_2(L_2)]' \cup [u_1(L_1) \wedge u_2(L_2)]'')'' \supseteq [u_1(L_1) \wedge u_2(L_2)]'' = [u_1(L_1) \cup u_2(L_2)]'' = L$$

by (i). Thus

$$[u_1(L_1') \wedge u_2(L'_2)]' = L$$

Taking the commutant once again we obtain

$$[u_1(L_1') \wedge u_2(L_2')]'' = L'. \qquad \blacksquare$$

Corollary 1. Let (L, u_1, u_2) be the product of L_1, L_2. Then L is irreducible only if L_1 and L_2 are irreducible.

Proof. If L is irreducible, then $L' = \{0, 1\}$. By Theorem 2, (iii) $u_1(L_1') \subset L'$, $u_2(L_2') \subset L'$, which implies that $L'_1 = L'_2 = \{0, 1\}$. ■

Corollary 2. The tensor product (L, u_1, u_2) is distributive iff L_1 and L_2 are distributive.

Proof. Let L_1 and L_2 be distributive, i.e., $L_i = L_i'$, $i = 1, 2$. From Theorem 2, (iii) we get $u_1(L_1') = u_1(L_1) \subset L'$, $u_2(L_2') = u_2(L_2) \subset L'$. As $u_1(L_1)$ and $u_2(L_2)$ generate L, we get $L' = L$. From this it follows that L is distributive. If L is distributive, then $L = L'$. For $i = 1, 2$, $u_i(L_i) \subset L = L'$ implies $u_i(L_i) \subset u_i(L_i)' \cap u_i(L_i) = u_i(L_i')$, i.e., $u_i(L_i) = u_i(L_i')$, which implies that $L_i = L_i'$. ■

Let (L, u_1, u_2) be a product of L_1 and L_2. For any $0 \neq a_2 \in L_2$ $(0 \neq a_1 \in L_1)$ the maps $u_{1, a_2(u_2, a_1)}$ defined by $u_{1, a_2}(a_1) = u_1(a_1) \wedge u_2(a_2)[u_{2, a_1}(a_2) = u_1(a_1) \wedge u_2(a_2)]$ are injective [see proof of Theorem 2, (ii)].

Corollary 3. Let L_1, L_2 be irreducible orthomodular σ lattices and (L, u_1, u_2) be their product. To any $c \in L'$, $c \neq 0, 1$, let there be $b_2 \in L_2$ (or $b_1 \in L_1$) such that u_{1, b_2} (or u_{2, b_1}) is surjective and $c \wedge u_2(b_2) \neq u_2(b_2), 0$ (or $c \wedge u_1(b_1) \neq u_1(b_1), 0$). Then L is irreducible.

Proof. Let $c \in L'$, $c \neq 0, 1$. The map u_{1, b_2} is injective and surjective, i.e., it is a bijection. Let $c_1 \in L_1$ be such that $u_{1, b_2}(c_1) = c \wedge u_2(b_2)$. Then

$u_{1,b_2}(c_1) \leftrightarrow u_{1,b_2}(L_1)$ implies $c_1 \leftrightarrow L_1, c_1 \neq 0, 1$, a contradiction with the irreducibility of L_1. ■

Remark 1. The statements of Theorem 2 are similar to that proved in Zecca (1968) by another definition of the tensor product.

Example 1. Let (X, S) be a measurable space, where S is a σ algebra of subsets of X, and let $\mathfrak{M}$ be a set of probability measures on S containing all measures μ_x concentrated on the points $x \in X$. Evidently, $\mathfrak{M}$ is quite full for S, and the Jauch–Piron property in the countable form is fulfilled. A quantum logic $(S, \mathfrak{M})$ of the type just described is called a classical logic (Gudder, 1970). Let $(S_i, \mathfrak{M}_i)$, $i = 1,2$, be two classical logics, where S_i is a σ algebra of subsets of a space X_i, $i = 1,2$. Let S be the product σ algebra on $X_1 \times X_2$. The set of all product measures $\mu_x \times \mu_y$, $x \in X_1$, $y \in X_2$, is quite full for S. Let us set

$$\alpha: S_1 \times S_2 \to S$$

$$E \times F \mapsto E \times F, \text{ i.e., } \alpha \text{ is the identity map}$$

$$\beta: \mathfrak{M}_1 \times \mathfrak{M}_2 \to \mathfrak{M}$$

$$(\mu_1, \mu_2) \mapsto \mu_1 \times \mu_2$$

where $\mathfrak{M} = \{\mu_1 \times \mu_2 : \mu_1 \in \mathfrak{M}_1, \mathfrak{M}_2 \in \mathfrak{M}_2\}$. Then

$$\beta(\mu_1, \mu_2)(\alpha(E_1, E_2)) = \mu_1 \times \mu_2(E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$$

and it can be easily checked that $(S, \mathfrak{M})_{\alpha,\beta}$ is the tensor product of $(S_1, \mathfrak{M}_1)$ and $(S_2, \mathfrak{M}_2)$ in the sense of Definition 2.

Example 2. Let us consider the case in which (L_1, M_1) is a quantum logic and (L_2, M_2) is a classical logic. This case is important from the physical point of view: it describes measurements of quantum observables by classical measurement devices. Let us set

$$\varphi_1: L_1 \to L, \qquad \varphi_2: L_2 \to L$$

$$a \mapsto \alpha(a, 1) \qquad b \mapsto \alpha(1, b)$$

where $(L, M)_{\alpha,\beta}$ is the tensor product of (L_i, M_i), $i = 1,2$. By Theorem 2, (iii), $\varphi_2(L_2) = \varphi_2(L_2') \subset L'$, where L' is the center of L. If $\{b_i\}_{i=1}^{\infty}$ is any partition of identity in L_2, then $\{\varphi_2(b_i)\}_{i=1}^{\infty}$ is the partition of identity in L'. Then L can be written as a direct sum $L = \oplus_{i=1}^{\infty} L_{[0, \varphi_2(b_i)]}$, and the logics

$L_{[0, \varphi_2(b_i)]}$ are irreducible iff $\varphi_2(b_i)$, $i = 1, 2, \ldots$ are atoms in L'. If the maps

$$\varphi_{2, b_i}: L_1 \to L$$
$$a \mapsto \varphi_1(a) \wedge \varphi_2(b_i)$$

are surjective, then the logics $L_{[0, \varphi_2(b_i)]}$ are isomorphic with L_1, so that L can be written as the direct sum of the copies of L_1 indexed by the set $\{b_i\}_{i=1}^{\infty}$.

4. TENSOR PRODUCT OF COMPLETE ATOMISTIC LATTICES

We shall consider quantum logics (L, M), where L is a complete atomistic lattice and M is a set of pure states such that to any atom $e \in L$ there is exactly one state $p \in M$ for which $p(e) = 1$. From the Jauch–Piron property we get that for any $p \in M$, $\{a: p(a) = 1\} = \{a: e \leqslant a\}$, where e is the atom such that $p(e) = 1$. The Jauch–Piron property is then fulfilled not only for countable sets, but for any sets. Clearly, M is quite full for L.

Theorem 3. Let (L_1, M_1) and (L_2, M_2) be two quantum logics such that L_1 and L_2 are complete atomistic orthomodular lattices and M_1 and M_2 are sets of pure states such that to any atom $e_1 \in L_1$ $(e_2 \in L_2)$ there is exactly one state $p_1 \in M_1$ $(p_2 \in M_2)$ such that $p_1(e_1) = 1$ $[p_2(e_2) = 1]$. Let α: $L_1 \to L_2$ and β: $M_1 \to M_2$ be mappings such that

(i) $\beta(m)(\alpha(a)) = m(a)$ for all $a \in L_1, m \in M_1$

(ii) β is onto

Then α and β are bijections.

Proof. Let $\beta(m_1) = \beta(m_2)$, then $\beta(m_1)(\alpha(a)) = \beta(m_2)(\alpha(a))$ for any $a \in L_1$, i.e., $m_1(a) = m_2(a)$ for any $a \in L_1$. Hence $m_1 = m_2$. Thus β is one-to-one.

Now $\beta(m)(\alpha(a^\perp)) = m(a^\perp) = 1 - m(a) = 1 - \beta(m)(\alpha(a)) = \beta(m)(\alpha(a)^\perp)$ for all $\beta(m) \in M_2$, and as $\beta[M_1] = M_2$ and M_2 is quite full, we have $\alpha(a^\perp) = \alpha(a)^\perp$.

From the Jauch–Piron property we obtain that for any index set I, $\beta(m)(\wedge_{i \in I} \alpha(a_i)) = 1 \Leftrightarrow \beta(m)(\alpha(a_i)) = 1$ for all $i \in I \Leftrightarrow m(a_i) = 1$ for all $i \in I \Leftrightarrow m(\wedge_{i \in I} a_i) = 1 \Leftrightarrow \beta(m)(\alpha(\wedge_{i \in I} a_i)) = 1$ for $\beta(m) \in M_2$, hence $\alpha(\wedge_{i \in I} a_i) = \wedge_{i \in I} \alpha(a_i)$.

From $\beta(m)(\alpha(1)) = m(1) = 1$ for all $\beta(m)$ we get $\alpha(1) = 1$. Thus we have shown that α is an orthohomomorphism.

If $\alpha(a)=\alpha(b)$, then $\beta(m)(\alpha(a))=\beta(m)(\alpha(b))$ implies $m(a)=m(b)$ for all $m\in M_1$, so that $a=b$. Hence α is one-to-one.

Let $A_i\subset L_i$ be the set of all atoms in L_i, $i=1,2$. Let s_i: $M_i\to A_i$, $i=1,2$ be such that $m_i(s_i(m_i))=1$. Let $a\in A_1$. If $\alpha(a)\notin A_2$, then there are $e_1, e_2\in A_2, e_1, e_2\leqslant\alpha(a)$. Let $q_1=s_2^{-1}(e_1), q_2=s_2^{-1}(e_2)$ and let $q_1=\beta(p_1), q_2=\beta(p_2)$, $p_1, p_2\in M_1$. Then $q_1(\alpha(a))=q_2(\alpha(a))=1$ implies $p_1(a)=p_2(a)=1$, but this implies that $p_1=p_2$. Hence $e_1=s_2\circ\beta(p_1)=s_2\circ\beta(p_2)=e_2$, i.e., $\alpha(a)\in A_2$. For $p\in M_1$, $p(s_1(p))=1$ implies that $\beta(p)(\alpha(s_1(p)))=1$, i.e., $s_2\circ\beta=\alpha\circ s_1$. Let α_{A_1} be α restricted to A_1. Then $\alpha_{A_1}: A_1\to A_2$ and $\alpha_{A_1}=s_2\circ\beta\circ s_1^{-1}$. As s_1, s_2 and β are bijections, α_{A_1} is also a bijection.

Let $c\in L_2$. Then $c=\vee\{c_i: c_i\in A_2, c_i\leqslant c\}=\vee\{\alpha_{A_1}(\alpha_{A_1}^{-1}(c_i): c_i\leqslant c\}=\alpha(\{\vee\,\alpha_{A_1}^{-1}(c_i): c_i\leqslant c\})$, i.e., α is onto. We have shown that α is an isomorphism. ■

Theorem 4. Let (L_1, M_1), (L_2, M_2), and (L, M) be quantum logics with the properties described in Theorem 3. Let $(L, M)_{\alpha,\beta}$ be the tensor product of (L_1, M_1) and (L_2, M_2). Then the maps

$$\varphi_{2,b}: L_1\to L_{[0,\varphi_2(b)]}$$

$$a\mapsto\alpha(a,b)$$

are bijections for any atom $b\in L_2$.

Proof. Let us consider the maps

$$\varphi_{2,b}: L_1\to L_{[0,\varphi_2(b)]}$$

$$a\mapsto\alpha(a,b)$$

and

$$\beta_q: M_1\to\beta[M_1\times\{q\}]$$

$$p\mapsto\beta(p,q)$$

where $q\in M_2$ is such that $q(b)=1$. Let $c_1, c_2\in L_{[0,\varphi_2(b)]}$. From the fact that $\beta[M_1\times M_2]$ is quite full, we obtain

$$\beta(m_1, m_2)(c_1)=1\Rightarrow\beta(m_1, m_2)(c_2)=1 \text{ implies } c_1\leqslant c_2$$

But $c_1, c_2\leqslant\varphi_2(b)$, so that

$$\beta(m_1, m_2)(c_1)=1\Rightarrow(m_1, m_2)(\varphi_2(b))=1$$

i.e., $(m_1, m_2)(\alpha(1, b)) = 1$, hence $m_2(b) = 1$. As b is an atom, $m_2 = q$. From this we see that

$$\beta(m_1, q)(c_1) = 1 \Rightarrow \beta(m_1, q)(c_2) = 1 \text{ implies } c_1 \leqslant c_2$$

i.e., the set $\beta[M_1 \times \{q\}]$ is quite full for $L_{[0, \varphi_2(b)]}$. As the map $\beta_q : M_1 \to \beta[M_1 \times \{q\}]$ is onto, it follows from Theorem 3 that $\varphi_{2, b}$ is a bijection. ■

Corollary 4. The map

$$\varphi_{1,a} : L_2 \to L_{[0, \varphi_1(a)]}$$

$$b \mapsto \alpha(a, b)$$

is a bijection for any atom $a \in L_1$.

Remark 2. If $L(H)$ is the logic of all closed subspaces of the Hilbert space H (complex, separable, $\dim H \geqq 3$), a set of states M is quite full for $L(H)$ iff it contains all the pure states (see Dvurečenskij and Pulmannová, 1980). Let $L_1(H_1)$ and $L_2(H_2)$ be two Hilbert space logics and let us look for their tensor product. It is natural to put $\alpha(P_1, P_2) = P_1 \otimes P_2$, $P_1 \in L_1$, $P_2 \in L_2$ and $\beta(\varphi_1, \varphi_2) = \varphi_1 \otimes \varphi_2$, $\varphi_1 \in H_1$, $\varphi_2 \in H_2$. But $L(H_1 \otimes H_2)$ (as well as $L(\overline{H}_1 \otimes H_2)$) cannot be a tensor product in the sense of Definition 2, because for the normed superposition $\sum_i c_i \varphi_i \times \psi_i$, $\varphi_i \in H_1$, $\psi_i \in H_2$, the corresponding state is not contained in $\beta[M_1 \times M_2]$, so that the set $\beta[M_1 \times M_2]$ is not quite full.

It depends on the physical nature of the considered physical systems, if the coupled system can be described by a tensor product in the sense of Definition 2 (or Definition 1), or if there should be put some additional conditions (e.g., the superposition principle).

Definition 2 could give a good mathematical description of the coupling of two physical systems in the case that at last one of the systems is a classical one, as it can be seen from the following section.

5. TENSOR PRODUCT OF ONE CLASSICAL AND ONE QUANTUM LOGICS

We recall that the direct sum $\oplus_{\alpha \in I} L_\alpha$ of a collection $\{L_\alpha : \alpha \in I\}$ of logics is the Cartesian product of the sets L_α endowed with the coordinatewise relation $\leqslant$ and unary operation $\perp$. That is, if $j = \{j_1, j_2, \ldots\}$ and $k = \{k_1, k_2, \ldots\}$ are elements of the product, then $j = k$ (respectively, $j^\perp = k$) iff $j_\alpha \leqslant k_\alpha$ (respectively, $j_\alpha^\perp = k_\alpha$) for any $\alpha \in I$.

Theorem 5. Let (L, M) be a quantum logic, where L is a τ lattice (τ is a cardinal). Let $(S, \mathfrak{M})$ be a classical logic, where S is the algebra of all subsets of X, card $X = \tau$. Then the quantum logic $(\tilde{L}, \tilde{M})$, where $\tilde{L} = \oplus_{x \in X} L_x$, $L_x = L$ for any $x \in X$, and $M = \{\delta_x \cdot m : m \in M, x \in X\}$, where $\delta_x \cdot m(\{a_y\}_{y \in X}) = m(a_x)$ is the tensor product of (L, M) and $(S, \mathfrak{M})$ in the category of τ logics.

Proof. First we show that $\tilde{M}$ is quite full for $\tilde{L}$. Let $a, b \in \tilde{L}$, $a = \{a_x\}_{x \in X}$, $b = \{b_x\}_{x \in X}$, and let

$$\{p \in M : p(a) = 1\} \subset \{p \in M : p(b) = 1\}$$

For $p = \delta_x \cdot m$, $x \in X$, we get $m(a_x) = 1 \Rightarrow m(b_x) = 1$, $m \in M$, i.e., $a_x \leqslant b_x$. As this is fulfilled for any $x \in X$, we obtain $a \leqslant b$.

Let us define the mappings α, β as follows:

$$\alpha : L \times S \to \tilde{L}$$

$$(a, E) \mapsto \{a_x\}_{x \in X}, \qquad a_x = \begin{cases} a & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

$$\beta : M \times \mathfrak{M} \to \tilde{M}$$

$$(m, \mu_x) \mapsto \delta_x \cdot m$$

Then

$$\beta(m, \mu_y)(\alpha(a, E)) = \delta_y \cdot m(\{a_x\}) = \begin{cases} m(a) & \text{if } y \in E \\ 0 & \text{if } y \notin E \end{cases}$$

i.e., $\beta(m, \mu_y)(\alpha(a, E)) = m(a) \cdot \mu_y(E)$.

Clearly, $\beta[M \times \mathfrak{M}] = \tilde{M}$, and $\tilde{M}$ is quite full for $\tilde{L} \cdot \tilde{L}$ is generated by the elements $\alpha(a, \{x\})$, $a \in L$, $x \in X$. Hence $(\tilde{L}, \tilde{M})$ is the tensor product of (L, M) and $(S, \mathfrak{M})$. ■

A set I is said to be real measurable (or of real-measurable cardinality) if there exists a nontrivial σ-additive measure $\mu : 2^I \to \langle 0, 1 \rangle$ which vanishes at points.

In Maňasová and Pták (1981) there is proved the following statement.

Theorem 6. Let $\{L_\alpha : \alpha \in I\}$ be a collection of logics, I non-real-measurable. Let m be a state on $\oplus_{\alpha \in I} L_\alpha$. Then there exists a sequence $\{\alpha_n : n \in N \subset I\}$ and a partition of unity $\{p_{\alpha_n} : n \in N\}$ such that, for any $a = (a_1, a_2, \ldots) \in \oplus_{\alpha \in I} L_\alpha$,

$$m(a) = m(a_1, a_2, \ldots) = \sum_{n=1}^{\infty} p_{\alpha_n} m_{\alpha_n}(a_n)$$

where m_{α_n} is a state of L_n.

For $N \subset M$ put $\bar{N} = \{m \in M: N(a) = 1 \Rightarrow m(a) = 1\}$, where $N(a) = 1$ means that $m(a) = 1$ for all $m \in N$ [see Gudder, 1971].

Theorem 7. Let (L, M) be a quantum logic such that L is a τ lattice, τ is non-real-measurable cardinal, and let the Jauch–Piron property in τ form hold, i.e., $m(a_\alpha) = 1$, $\alpha \in I$, card $I = \tau$ implies that $m(\wedge_{\alpha \in I} a_\alpha) = 1$ for any $m \in M$. Further, let there be to any $N \subset M$ an element $a \in L$ such that $\bar{N} = \{m \in M: m(a) = 1\}$. Let $(S, \mathfrak{M})$ be a classical logic such that S is the algebra of all subsets of X, card $X = \tau$. Then if $(\tilde{L}, \tilde{M})_{\alpha, \beta}$ is a tensor product of (L, M) and $(S, \mathfrak{M})$, then $\tilde{L} = \oplus_{x \in X} L_x$, $L_x = L$, and $\tilde{M} = \{\delta_x \cdot m: x \in X, m \in M\}$.

Proof. Put

$$u_1: L \to \tilde{L}, \qquad u_2: S \to \tilde{L}$$
$$a \mapsto \alpha(a, X) \qquad E \mapsto \alpha(1, E)$$

It is easy to check that u_1, u_2 are τ homomorphisms. By Theorem 2, (iii), $u_2(S) \subset \tilde{L}'$, so that $u_2(\{x\}) \in \tilde{L}'$ for any $x \in X$. Then $\tilde{L}$ can be written in the form $\tilde{L} = \oplus_{x \in X} \tilde{L}_{[0, u_2(\{x\})]}$. It can be shown as in the proof of Theorem 4, that the set $\beta[M \times \mu_x\}$ is quite full for $L_{[0, u_2(\{x\})]}$. Put

$$u_{1,x}: L \to \tilde{L}_{[0, u_2(\{x\})]}$$
$$a \mapsto \alpha(a, \{x\})$$

We show that $u_{1,x}$ is surjective. Let $c \in \tilde{L}_{[0, u_2(\{x\})]}$. Let us set

$$N = \{m \in M: \beta(m, \mu_x)(c) = 1\}$$

If $a \in L$ is the element such that $N = \{m \in M: m(a) = 1\}$, then $\{p \in [M \times \mu_x]: p(c) = 1\} = \{p \in \beta[M \times \mu_x]: p(\alpha(a, \{x\})) = 1\}$, i.e., $c = \alpha(a, \{x\}) = u_{1,x}(a)$.

Thus we have shown that $\tilde{L} = \oplus_{x \in X} L_x$, $L_x = L$, $x \in X$. By Theorem 6, any state $p \in \beta[M \times \mathfrak{M}]$ is of the form $p = \sum_{n=1}^{\infty} p_{\alpha_n} m_{\alpha_n}$. From $\beta(m, \mu_y)(\alpha(a, E)) = m(a)\mu_y(E)$ it follows that $\beta(m, \mu_y) = \delta_y \cdot m$. ∎

The representation of a tensor product in the form of the direct sum of copies of L indexed by X might be appropriate for describing quantum measurements; any of the copies L_x of L would correspond to some position on the scale of the measurement apparatus.

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